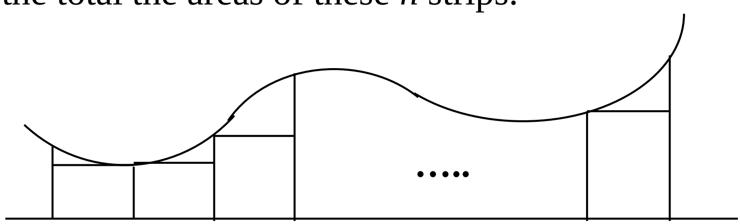
12. NUMERICAL INTEGRATION

§12.1 The Trapezium Rule

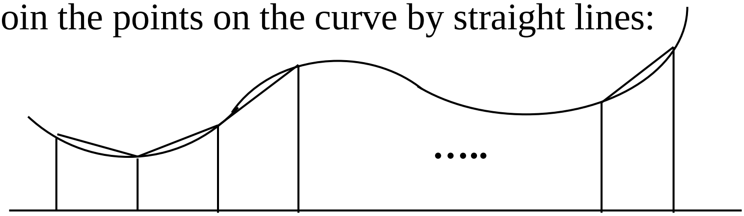
It often happens that we need to find the value of a definite integral for a function whose indefinite integral can't be found in terms of the elementary functions that we know about already. Since a definite integral represents an area between the graph of the function and the *x*-axis, all we need to do is to find some way of approximating this area.

Of course we could sketch the curve on graph paper and estimate the area by counting squares. But not only is this rather inconvenient, there's no way we could get a high degree of precision.

Let's suppose that we have a function $y = \mathrm{f}(x)$ and we wish to estimate $\int_a^b y \, \mathrm{d}x$. If we divide the interval from a to b into n strips of equal width h these strips are basically rectangles. We can easily find the area of each rectangle and the total the areas of these n strips.

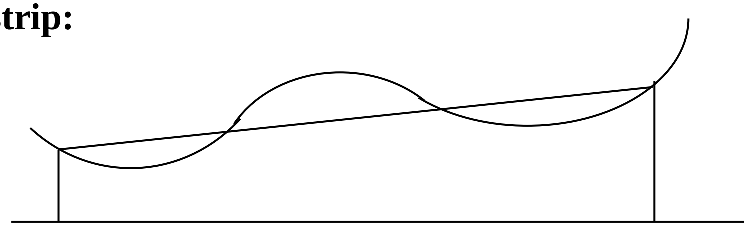


However a much better approximation can be obtained if we join the points on the curve by straight lines:

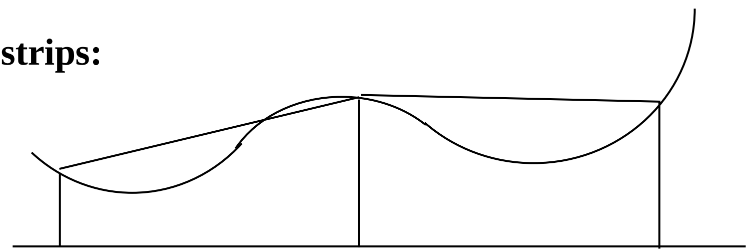


Sometimes the straight line goes above the curve and we include a little more area than we should. Sometimes it goes below and we underestimate the strip of area. On average these tend to balance each other – but not completely. We rarely get the exact value in this way. But the more strips we take the more closely the lines will follow the curve and the better the approximation.

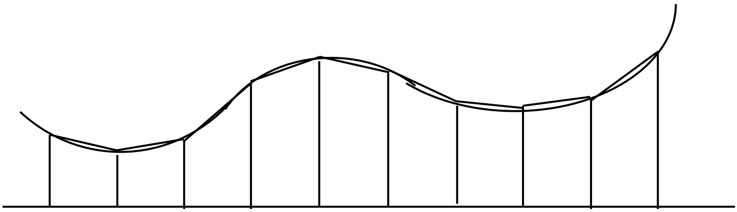
One strip:



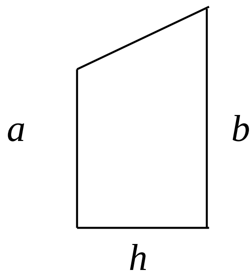
Two strips:



Nine strips:



The area of a trapezium is the average length of the parallel sides multiplied by the distance between them.



Area = $\left(\frac{a+b}{2}\right)h$.

Suppose we want to estimate $\int_a^b y \, dx$ and we use n strips, each of width h. Suppose the y-values at the end-points of the strips are $y_0, y_1, \ldots, y_n$.

The total area of the strips, approximating each of them by a trapezium, is:

$$\frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\,[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})].$$

Trapezium Rule

$$\int_a^b y\,dx \approx \frac{\text{width}}{2}\,[\text{First} + \text{Last} + 2 \times \text{Sum of others}]$$

Example 1: Estimate $\int_0^{10} x^2\,dx$ by the Trapezium Rule, using 5 strips.

Solution: The width of each strip is $w = 2$.

x	$y = x^2$
0	0
2	4
4	16
6	36
8	64
10	100

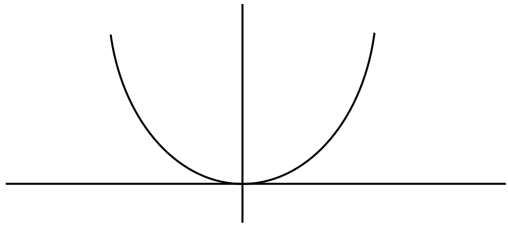
So $\int_0^{10} x^2\,dx \approx \frac{2}{2}.[0 + 100 + 2(4 + 16 + 36 + 64)] = 340.$

The exact value is $\int_0^{10} x^2\,dx = \left[\frac{1}{3}x^3\right]_0^{10} = \frac{1000}{3} \approx 333.3.$

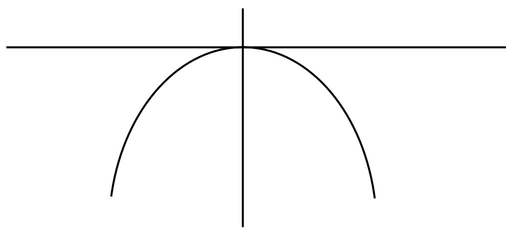
We can improve the accuracy of our estimate by fitting quadratic functions to our points rather than straight lines.

§12.2 Parabolas

A parabola is the shape of the graph of a quadratic. The basic parabola is $y = x^2$.



We can vary the shape by changing the coefficient of x^2 and adding an x terms and a constant term. The general equation for a parabola, with its axis parallel to the y-axis, is $y = ax^2 + bx + c$. This moves the parabola left or right and up or down, and changes the scale. If a is negative it turns the parabola upside down.



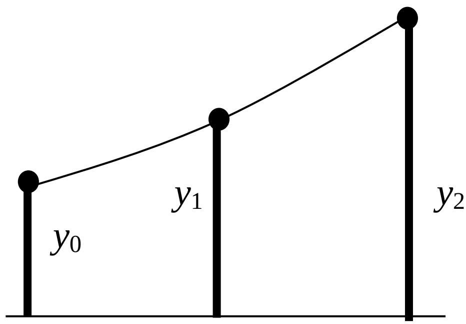
The parabola is a very useful shape. Parabolic mirrors are used as reflectors in car headlights and in reflective telescopes because they have the property of producing a parallel beam of light, or focussing a parallel beam at a single point. Parabolas are very useful in approximating other curves. While a parabola and some other curve may differ, over a small enough interval the approximation can be quite good. What we do therefore is to break up an arbitrary curve into small sections and approximate each section by a different parabola.

And why might we wish to approximate a general curve by a series of parabolas? Well, so that we can estimate the area under the curve for functions that we can't integrate.

§12.3 Area Under a Parabola

We're going to develop a formula for the area under a parabola (strictly speaking, between the parabola, the x-axis and two vertical lines) in terms of just 3 points on that parabola.

Let's take the parabola $y = x^2$ and a point at $x = u$ on it. Now let's move a small distance to the left and the same distance to the right, giving two other points at $x = u - h$ and $x = u + h$. So we have three points, equally spaced horizontally, a distance w apart. Let's write the corresponding
y-values as y_0, y_1 and y_2.
So:
$y_0 = (u - h)^2$,
$y_1 = u^2$ and
$y_2 = (u + h)^2$.



The area of these two strips is $\int_{u-h}^{u+h} x^2 \, dx$

$$= \left[\frac{1}{3}x^3\right]_{u-h}^{u+h} = \frac{1}{3}\left[(u+h)^3 - (u-h)^3\right] = \frac{h}{3}\left[6u^2 + 2h^2\right].$$

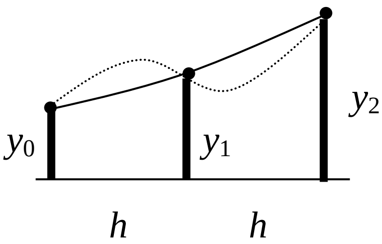
Since $y_0 + y_2 = (u - h)^2 + (u + h)^2 = 2h^2 + 2u^2$ and $y_1 = u^2$

we have $\int_{u-h}^{u+h} x^2 \, dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$.

So $\int_a^b (\text{any quadratic}) = \frac{h}{3}[y_0 + 4y_1 + y_2]$ where $h = \frac{b-a}{2}$ is the width of the 2 strips and y_0, y_1, y_2 are the y-values of the three endpoints of these two strips.

§12.4 Simpson's Rule

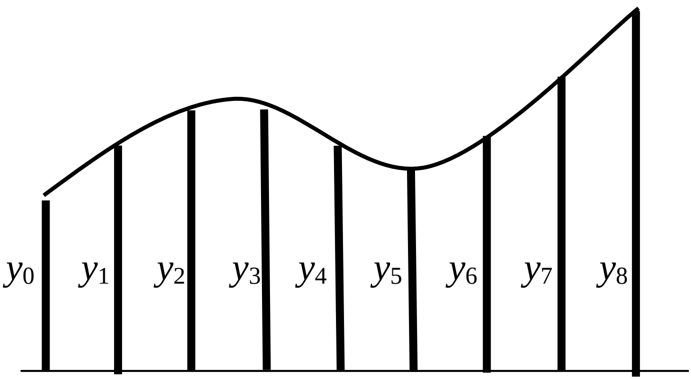
The above formula is exact for quadratics. But if we approximate another function by a quadratic we can use the same formula to approximate the area for that other function.



The dotted graph represents some function $y = f(x)$ which we've approximated by the quadratic that passes through the three points shown. The solid curve represents that quadratic. The area under the dotted curve will be approximately equal to that under the quadratic, so the formula $\frac{h}{3}[y_0 + 4y_1 + y_2]$ will give an estimate for the area under $y = f(x)$.

The closeness of the approximation depends on how closely the quadratic follows the given curve. If the strips are narrow the approximation will be very good. The narrower the strips, the better the approximation.

The trouble is that if we want the area from a to b, the width $h = ½ (b – a)$ may not be very small. The trick is to divide the interval $[a, b]$ into an even number of strips and *fit a different quadratic* to each pair of strips.



The y-values of all of these points are called **ordinates** and if we denote them by $y_0, y_1, y_2, y_3, \ldots, y_{2n}$ (where there are $2n$ strips) then we can use the basic $\frac{h}{3}[y_0 + 4y_1 + y_2]$ formula for each pair of strips and add the results together.

For the first pair of strips we can use the formula exactly, to give $\frac{h}{3}[y_0 + 4y_1 + y_2]$.
For the second pair of strips we have ordinates y_2, y_3, y_4 and so the area becomes $\frac{h}{3}[y_2 + 4y_3 + y_4]$.

The total area for 8 strips is therefore (the spacing puts the same y_i's underneath one another) is $\frac{h}{3}$ times the following:

$$\begin{aligned}&[y_0 + 4y_1 + y_2]\\ &\quad + [y_2 + 4y_3 + y_4]\\ &\qquad + [y_4 + 4y_5 + y_6]\\ &\qquad\quad + [y_6 + 4y_7 + y_8]\end{aligned}$$

$$= \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8].$$

A similar formula holds for any (even) number of strips. Never attempt to use an odd number of strips. The easiest way to remember this is to call ordinates **odd** or **even** according to the subscript. So y_0, y_2, .. are the even ordinates and y_1, y_3, … are the odd ones. Notice that the odd ordinates have a weighting of 4 in the formula while the even ones have a weighting of 2, except for the first and last which, while even, only have a weighting of 1. **Simpson's Rule**, can thus be expressed as:

$$\int_a^b (\textbf{any function}) \approx \frac{\textbf{width}}{3}\left[\textbf{first} + \textbf{last} + 2(\textbf{sum of other evens}) + 4(\textbf{sum of odds})\right]$$

§12.5 Simpson's Spreadsheet

Like Newton's Method, Simpson's Rule is best done in a spreadsheet, even if you are doing it by hand with the aid of a calculator. Setting the working out in table form makes for fewer errors.

Like Newton's Spreadsheet, this one has 4 columns. The headings are ***x***, ***y***, ***w*** and ***wy***. The ***w***'s are the **weights**. These are 1's 2's and 4's as appropriate. The ***wy*** column contains the product of the ***w***'s and the ***y***'s. The ***y***'s are the ordinates, got by substituting the ***x***'s into the function. And the ***x***'s are evenly spaced over the interval over which we're integrating.

x	***y***	***w***	***wy***

The first thing to do is to decide how many strips you're going to use. You must use an even number of strips. The more strips you use the more work you'll have to do. But, up to a point, the more strips the more accurate will be the answer, but not always. We'll discuss the number of strips you should use later.

Having decided on the number of strips, you work out the width of each. If you're integrating from a to b and have $2n$ strips then the width is $h = \frac{b - a}{2n}$.

In the first row, in the ***x*** column, you put down the bottom limit of integration, a. You then add h to each x to get the next, and keep stepping out until you reach the top limit, b. With $2n$ strips this should give you $2n + 1$ rows. (There's always one more endpoint than strips.)

x	***y***	***w***	***wy***
a			
$a + h$			
$a + 2h$			
$a + 3h$			
…..			
b			

The next step is to substitute each of these values of x into the function and write down the corresponding y-values into the ***y***-column.

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{w}$	$\boldsymbol{wy}$
a	y_0		
$a + h$	y_1		
$a + 2h$	y_2		
$a + 3h$	y_3		
…..	…..		
b	y_{2n}		

The weights always follow the same pattern. The first is 1, and then they alternate 4, 2, 4, 2, … until the second last is a 4 and then the very last is a 1.

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{w}$	$\boldsymbol{wy}$
a	y_0	1	
$a + h$	y_1	4	
$a + 2h$	y_2	2	
$a + 3h$	y_3	4	
…..	…..	…..	
b	y_{2n}	1	

The ***wy*** column is now computed by multiplying each y-value by the appropriate weight:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{w}$	$\boldsymbol{wy}$
a	y_0	1	y_0
$a + h$	y_1	4	$4y_1$
$a + 2h$	y_2	2	$2y_2$
$a + 3h$	y_3	4	$4y_3$
…..	…..	…..	.....
b	y_{2n}	1	y_{2n}

You then total the ***wy*** column.

x	***y***	***w***	***wy***
a	y_0	1	y_0
$a + h$	y_1	4	$4y_1$
$a + 2h$	y_2	2	$2y_2$
$a + 3h$	y_3	4	$4y_3$
…..	…..	…..	.....
b	y_{2n}	1	y_{2n}
TOTAL			T

Underneath this you write **INTEGRAL** and then multiply T by $h/3$:

x	***y***	***w***	***wy***
a	y_0	1	y_0
$a + h$	y_1	4	$4y_1$
$a + 2h$	y_2	2	$2y_2$
$a + 3h$	y_3	4	$4y_3$
…..	…..	…..	.....
b	y_{2n}	1	y_{2n}
TOTAL			T
INTEGRAL			$h/3 \times \text{T}$

Example 2: Use Simpson's Rule to approximate $\int_1^4 \sqrt{x}\, dx$ using 6 strips. Compare this with the exact value.
Solution: The width is $h = 0.5$

x	*y*	*w*	*wy*
1	1	1	1
1.5	1.2247	4	4.8988
2	1.4142	2	2.8284
2.5	1.5811	4	6.3244
3	1.7320	2	3.4640
3.5	1.8708	4	7.4832
4	2	1	2.0000
TOTAL			27.9988
INTEGRAL			4.6665

Performing the exact integration we get:

$$\int_1^4 \sqrt{x}\,\mathrm{d}x = \int_1^4 x^{1/2}\,\mathrm{d}x = \left[\frac{2}{3}x^{3/2}\right]_1^4$$

$$= \frac{2}{3}\,[4^{3/2} - 1^{3/2}] = \frac{2}{3}\,[2^3 - 1] = \frac{14}{3} \approx 4.6667$$

So even with as few strips as 6 we've achieved a very good approximation by Simpson's Rule.

§12.6 How Many Strips?

Remember:

You must use an even number of strips with Simpson's Rule.

Secondly, if you use Simpson's Rule to integrate a quadratic you need only two strips. Naturally if you approximate a quadratic by a quadratic you'll come up with the exact value, no matter how many strips you use. So in this case you may as well use as few strips as possible, that is, two strips.

When integrating a cubic Simpson's Rule happens to be exact, even with as few as 2 strips. Although the quadratic approximation doesn't match the cubic, the bits where the quadratic goes above the cubic are compensated exactly by the bits where it goes underneath.

Of course there's no reason why you should be attempting to use Simpson's rule for a quadratic, or a cubic or any function that you can integrate easily. In the above example we used Simpson's Rule to integrate $\sqrt{x}$ from 1 to 4. But we can integrate $\sqrt{x}$ and this will give the exact value with much less work.

Don't use Simpson's Rule if you can integrate the function easily.

Usually Simpson's Rule is not exact. But the more strips you take (only ever an even number) the more accurate the answer, at least in theory.

In practice there's the phenomenon of round-off errors. Your calculator will calculate the ordinates to so many decimal places but there are usually tiny errors for each ordinate. If you were to take an enormous number of strips, these round-off errors might very well build up and counteract the better accuracy because the parabolas are fitting better.

Example 3: I used Simpson's Rule to estimate $\int_1^4 \sqrt{x}\, dx$ using more and more strips. Keep in mind that the exact value is 4.66666….

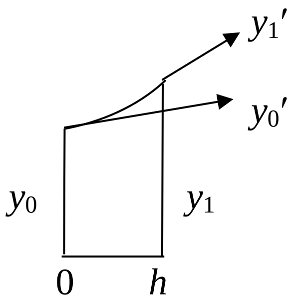
# strips	**estimate**	**# places of accuracy**
2	4.662277660	2
4	4.666220710	3
6	4.666563053	3
8	4.666631374	4
10	4.666651629	4
20	4.666665668	5
30	4.666666467	6
50	4.666666642	7
100	4.666666668	8
1000	4.666666656	7

You'll see from this that you get quite a lot of accuracy with relatively few strips. A good rule of thumb, when using Simpson's Rule with hand calculation, is to use 6, 8 or 10 strips. Your decision as to how many strips you choose may be based on convenience as much as accuracy. For example integrating from 1 to 4 using 6 strips gives a width of 0.5 while 8 strips gives a width of 0.375 which is less convenient. Ten strips may be a good choice in that it combines convenience with a fairly good degree of accuracy.

But small improvements in accuracy come at a very great price in amount of computation. If you were doing the calculations on a computer you might decide that 1000 strips would be your choice. But notice that, working to 9 decimal places, gave a little less accuracy with 1000 strips than what was achieved with only 100. More is not always better.

§12.7 The Cubic Fit Method

Why don't we fit a cubic $y = ax^3 + bx^2 + cx + d$ to each strip? This requires four pieces of information to enable us to find the four coefficients. We could take y_0 and y_1, the ordinates at the end-points, as well as y_0' and y_1', the slope at these points.



In general a cubic would be able to fit a given curve more closely than a quadratic and so give a more accurate estimate of the area. The amazing thing is that the Cubic Fit Rule, approximating the curve on each strip by a cubic, is precisely the Trapezium Rule with a correction factor. Moreover this correction factor involves very little work in that it only involves the two derivatives at each end of the whole interval. Yet, on the whole, it gives a much better estimate for the same number of strips.

Cubic Fit Rule

$$\int_a^b y \, dx \approx \underbrace{\frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]}_{\text{Trapezium Rule}} - \underbrace{\frac{h^2}{12}[y']_a^b}_{\text{Correction}}$$

Here $[y']_a^b$ is y' at $x = b$ minus y' at $x = a$.

I used the Trapezium Rule, Simpson's Rule and the Cubic Fit Rule, each with 4 strips to estimate $\int_1^5 \sqrt{x}\, dx$ and with 6 strips to estimate $\int_2^8 \log x \, dx$. I then compared them with the exact values and worked out the percentage errors.

The results were:

	$\sqrt{x}$	log x
Trapezium Rule	0.3%	0.3344%
Simpson's Rule	0.02%	0.0114%
Cubic Fit Method	0.004%	0.0035%

I must point out that occasionally the Cubic Fit Method performs worse than Simpson's and, of course you'd never use the Cubic Fit Method if you had difficulty in finding the derivative.

My advice is to treat the Cubic Fit Method as a curiosity and to use Simpson's Rule. After all, if you're using a computer spreadsheet a few extra strips in Simpson's Rule can give you all the accuracy you need, and with copying formulae in the spreadsheet it's not really that much more work to have more strips. Anyway, if you're interested in seeing how the Cubic Fit Formula is derived, you can consult my notes *Techniques of Calculus*. But, personally, I wouldn't bother!

EXERCISES FOR CHAPTER 12

Use The Trapezium Rule, Simpson's Rule and the Cubic Fit Method, each with 8 strips, to approximate the following definite integrals. Work to 4 decimal places. Then work out the percentage error with each of these methods. The exact value is given.

Exercise 1: $\int_0^8 x^2 \, dx = 170.6667$

Exercise 2: $\int_1^9 \sqrt{x} \, dx = 17.3333$

Exercise 3: $\int_1^5 \frac{dx}{x} = 1.6094$

Exercise 4: $\int_2^4 e^{x^2} \, dx = 1.14938 \times 10^6$

Exercise 5: $\int_{10}^{30} \log x \, dx = 59.0101$

Exercise 6: $\int_0^2 2^x \, dx = 4.3281$ [**HINT:** $2^x = e^{(\log 2)x}$]

Exercise 7: $\int_0^1 \frac{1}{1 + x^2} \, dx = 0.7854$

Exercise 8: $\int_1^{17} x^3 - 8\sqrt{x} + 2 \, dx = 20511.5051$

In the following exercises use the 'best' method to approximate the given definite integrals. Comment on the accuracy of your answers.

Exercise 9: $\int_1^8 x^3 - x^2 + 7 \, dx$

Exercise 10: $\int_0^{12} \sqrt{x^3 + 1} \, dx$

Exercise 11: $\int_1^4 \sqrt{x^2 - x} \, dx$

Exercise 12: $\int_0^6 f(x)\,dx$ where $f(x)$ fits the following table of values.

x	0	1	2	3	4	5	6
$f(x)$	2	3	5	5	4	2	1

SOLUTIONS FOR CHAPTER 12

Exercise 1:

TRAPEZIUM		SIMPSON	
x	***y***	***w***	***wy***
0	0	1	0
1	1	4	4
2	4	2	8
3	9	4	36
4	16	2	32
5	25	4	100
6	36	2	72
7	49	4	196
8	64	1	64
$\int$	**172**	$\int$	**170.6667**

$y' = 2x$ so Correction $= \frac{1}{12}[16 - 0] = 1.3333$

TRAPEZIUM	172.0000	0.8%
SIMPSON	170.6667	0%
CUBIC FIT	170.6667	0%
EXACT VALUE	170.6667	

Exercise 2:

TRAPEZIUM

x	***y***
1	1
2	1.4142
3	1.7321
4	2
5	2.2361
6	2.4495
7	2.6458
8	2.8284
9	3
∫	**17.3061**

SIMPSON

w	***wy***
1	1
4	5.6568
2	3.4642
4	8
2	4.4722
4	9.798
2	5.2916
4	11.3136
1	3
∫	**17.3321**

$y' = \frac{1}{2\sqrt{x}}$ so Correction = $\frac{1}{12}$ [0.1667 – 0.5] = – 0.0278

TRAPEZIUM	17.3061	0.16%
SIMPSON	17.3321	0.007%
CUBIC FIT	17.3339	0.003%
EXACT VALUE	17.3333	

Exercise 3:

TRAPEZIUM			**SIMPSON**	
x	***y***		***w***	***wy***
1	1		1	1
1.5	0.6667		4	2.6668
2	0.5		2	1
2.5	0.4		4	1.6
3	0.3333		2	0.6666
3.5	0.2857		4	1.1428
4	0.25		2	0.5
4.5	0.2222		4	0.8888
5	0.2		1	0.2
$\int$	**1.6289**		$\int$	**1.6108**

$y' = -\frac{1}{x^2}$ so Correction $= \frac{0.25}{12}[-0.04 + 1] = 0.02$

TRAPEZIUM	1.6289	1.2%
SIMPSON	1.6108	0.09%
CUBIC FIT	1.6089	0.03%
EXACT VALUE	1.6094	

Exercise 4:

TRAPEZIUM			**SIMPSON**	
x	***y***		***w***	***wy***
2	54.5982		1	54.5982
2.25	157.9850		4	631.9400
2.5	518.0128		2	1036.0256

2.75	1924.6511		4	7698.6044
3	8103.0839		2	16206.1678
3.25	38657.6514		4	154630.6056
3.5	208981.2889		2	417962.5778
3.75	1280165.5968		4	5120662.3872
4	8886110.5205		1	8886110.5205
∫	**1495397.7073**		∫	**1217082.7856**

$y' = 2xe^{x^2}$ so Correction

$$= \frac{0.25^2}{12}[71088884.1641 - 218.3926] = 370253.4676$$

TRAPEZIUM	1.49540×10^6	30%
SIMPSON	1.21708×10^6	6%
CUBIC FIT	1.12514×10^6	2%
EXACT VALUE	1.14938×10^6	

Exercise 5:

TRAPEZIUM			**SIMPSON**	
x	***y***		***w***	***wy***
10	2.3026		1	2.3026
12.5	2.5257		4	10.1028
15	2.7080		2	5.4160
17.5	2.8622		4	11.4488
20	2.9957		2	5.9914
22.5	3.1135		4	12.4540
25	3.2189		2	6.4378

27.5	3.3142		4	13.2568
30	3.4012		1	3.4012
∫	**58.9753**		∫	**59.0095**

$y' = \dfrac{1}{x}$ so Correction $= \dfrac{2.5^2}{12}[0.0333 - 0.1] = -0.0347$

TRAPEZIUM	58.9753	0.06%
SIMPSON	59.0095	0.001%
CUBIC FIT	59.0100	0.0002%
EXACT VALUE	59.0101	

Exercise 6:

TRAPEZIUM			**SIMPSON**	
x	***y***		***w***	***wy***
0	1		1	1
0.25	1.1892		4	4.7568
0.5	1.4142		2	2.8284
0.75	1.6818		4	6.7272
1	2		2	4
1.25	2.3784		4	9.5136
1.5	2.8284		2	5.6568
1.75	3.3636		4	13.4544
2	4		1	4
∫	**4.3389**		∫	**4.3281**

$y' = \log 2.2^x$ so Correction $= \dfrac{0.25^2}{12}[2.7726 - 0.6931]$

$= 0.0108$

TRAPEZIUM	4.3389	0.12%
SIMPSON	4.3281	0%
CUBIC FIT	4.3281	0%
EXACT VALUE	4.3281	

Exercise 7:

TRAPEZIUM		SIMPSON	
x	***y***	***w***	***wy***
0	1	1	1
0.125	0.9846	4	3.9384
0.25	0.9412	2	1.8824
0.375	0.8767	4	3.5068
0.5	0.8	2	1.6
0.625	0.7191	4	2.8764
0.75	0.64	2	1.28
0.875	0.5664	4	2.2656
1	0.5	1	0.5
∫	**0.7847**	∫	**0.7854**

$y' = -\dfrac{2x}{(1+x^2)^2}$ so Correction $= \dfrac{0.125^2}{12}[-0.5+0]$

$= -0.0007$

TRAPEZIUM	0.7847	0.09%
SIMPSON	0.7854	0%
CUBIC FIT	0.7854	0%
EXACT VALUE	0.7854	

Exercise 8:

TRAPEZIUM

x	***y***
1	–5
3	15.1436
5	109.1115
7	323.8340
9	707
11	1306.4670
13	2170.1556
15	3346.0161
17	4882.0151
∫	**20832.4707**

SIMPSON

w	***wy***
1	–5
4	60.5744
2	218.2230
4	1295.336
2	1414
4	5225.8680
2	4340.3112
4	13384.0644
1	4882.0151
∫	**20543.5947**

$y' = 3x^2 - \frac{4}{\sqrt{x}}$ so Correction $= \frac{2^2}{12}$ [866.0299 + 1] = 289.01

TRAPEZIUM	20832.4707	1.6%
SIMPSON	20543.5947	0.16%
CUBIC FIT	20543.4607	0.16%
EXACT VALUE	20511.5051	

Exercise 9: Here it is not difficult to find the indefinite integral: $\frac{x^4}{4} - \frac{x^3}{3}$ + 7x, and evaluating it at the endpoints we find that the integral is:

909.33333 – 6.91667 = 902.4167, correct to 4 decimal places.

The Cubic Fit Method would be almost as easy to use. Clearly one strip is sufficient, seeing that we are fitting a cubic to a cubic. This will give the exact value.

The Trapezium Method will give $\frac{7(455-7)}{2} = 1617$ and the correction factor, with $y' = 3x^2 - 2x$, is:

$$\frac{49(176-2)}{12} = 714.58333.$$

The Cubic Fit Method thus gives, correct to 4 decimal places, 1617 – 714.58333 = 902.4167,

Simpson's Rule would also be exact, but we would have to use 2 strips, not 1.

Exercise 10: This is not easy to integrate so we shall use a numerical method. We will use the Cubic Fit Method:

TRAPEZIUM

x	***y***
0	1
4	8.06226
8	22.64950
12	41.58125
$\int$	**208.00954**

$y' = \dfrac{3x^2}{2\sqrt{x^3 + 1}}$ so the Cubic Correction is:

$$\frac{16}{12}[5.19465 - 0] = 6.92620.$$

Hence the Cubic Fit gives 208.00954 – 6.92620

= 201.1033 as the estimate.

But is 201 correct to the nearest integer?

Let's try it with 6 strips.

TRAPEZIUM

x	***y***
0	1
2	3
4	8.06226
6	14.73092
8	22.64950
10	31.6386
12	41.58125
$\int$	**202.74381**

The Correction this time is $\dfrac{6.9260}{4}$ = 1.73155 since h is now 2, not 4. The Cubic Fit Method gives:

$$202.74381 - 1.73155 = 201.0123.$$

Since the improvement is only in the first decimal place it is reasonable to expect that both are correct to the nearest integer, namely 201.

Exercise 11: It is clearly too hard to find the indefinite integral, so a numerical method is called for. But before we embark on the Cubic Fit Method, consider the Correction Factor.

$y' = \frac{2x}{2\sqrt{x^2 - x}} = \frac{x}{\sqrt{x^2 - 1}}$. We will need to evaluate this at the endpoints and there is a problem at $x = 1$, the bottom end-point. This is one of the limitations of the Cubic Fit Method. If the derivative is zero at either end-point we cannot use this method. We must fall back on Simpson's Rule. We'll use 6 strips to start with.

SIMPSON

x	***y***	***w***	***wy***
1	0	1	0
1.5	0.86603	4	3.46412
2	1.42421	2	2.8242
2.5	1.93649	4	7.74596
3	2.44949	2	4.89898
3.5	2.95804	4	11.83216
4	3.46410	1	3.46410
		$\int$	**5.70562**

Before we can be reasonably certain as to how accurate this is, we need to repeat the process with more steps to see how much the estimate changes. Let's use 12 strips.

SIMPSON

x	***y***	***w***	***wy***
1	0	1	0
1.125	0.55902	4	2.23608
1.5	0.86603	2	1.73206

1.75	1.14564	4	4.46256
2	1.41421	2	2.82842
2.25	1.67705	4	6.70820
2.5	1.93649	2	3.87298
2.75	2.19374	4	8.77496
3	2.44949	2	4.89898
3.25	2.70416	4	10.81664
3.5	2.95804	2	5.91608
3.75	3.21131	4	12.84524
4	3.46410	1	3.46410
		$\int$	**5.73025**

It would appear that 5.7 is correct to one decimal place.

Exercise 12: Here we do not have a function, only some values. So we can't find the indefinite integral. But nor can we use the Cubic Fit Method, because we have nothing to differentiate. We can only use the Trapezium Rule, or Simpson's Rule and, as we know, Simpson's Rule is superior to the Trapezium Rule. We'll use 6 strips. (We can't use any more since we don't have the values at the intermediate points.)

SIMPSON

x	***y***	***w***	***wy***
0	2	1	2
1	3	4	12
2	5	2	10
3	5	4	20
4	4	2	8

5	2	4	8
6	1	1	1
		$\int$	**20.33**

It's hard to comment on the accuracy because we can't use any more strips and we don't have any more information to go by. From experience we could guess that it is accurate to the nearest integer, and possibly to one decimal place. So we might quote 20.3 as our estimate.